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Note

Convex Sets of Some Doubly Stochastic Matrices

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Let Ω denote the set of all n by n doubly stochastic matrices. Let t be a real number such that $1/t \geqslant 1/n$ and let m be a real number such that $1/m \leqslant 1 - 1/t$. The set $\Omega_s = \{A \in \Omega : 1/m \leqslant a_{ij} \leqslant 1/t,\ 1 \leqslant i, j \leqslant n\}$ is the convex hull of the matrices in Ω_s having as many largest entries, namely, $1/t$, as possible in each row and column while filling out the remaining entries with the value $1/m$ and if necessary at most one entry in each row and column which has a value between $1/m$ and $1/t$.

Ω shall denote the set of all n by n doubly stochastic matrices. Watkins and Merris [1] proved that the set of all doubly stochastic matrices with all of their entries less than or equal to $1/m$, where m is a positive integer, is the convex hull of the doubly stochastic matrices with m entries in each row and column equal to $1/m$ and the others equal to 0. The following theorem is an extension of the one they proved.

THEOREM. *Let t be a real number such that $1/t \geqslant 1/n$, and let m be a real number such that $1/m \leqslant 1 - t$. The set $\Omega_s = \{A \in \Omega: 1/m \leqslant a_{ij} \leqslant 1/t, 1 \leqslant i, j \leqslant n\}$ is the convex hull of the matrices in Ω_s having as many largest entries, namely, $1/t$, as possible in each row and column while filling out any other positions with the minimum value $1/m$ and if necessary at most one entry in each row and column which is between $1/m$ and $1/t$.*

Proof. We shall call an entry which is between $1/m$ and $1/t$ a middle entry. Let $S \in \Omega_s$ and let S have at least one row which contains two middle entries; then S must be a proper convex combination of two other matrices in Ω_s. This part of the proof parallels that done by Watkins and Merris [1].

Suppose that row i has two middle entries; then in each column corre-

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sponding to these middle entries there must be two middle entries. S has a row or column which contains two middle entries, so we shall choose a cycle of such entries, (i_1, j_1), (i_1, j_2), (i_2, j_2),..., (i_r, j_r), (i_r, j_1). Let E_1 be the matrix obtained by adding ϵ to the positions (i_k, j_k) where $1 \leq k \leq r$, and subtracting ϵ from the other entries of the cycle. Let $E_2 = 2S - E_1$ and make ϵ small enough such that E_1 and E_2 are in Ω_s. Then $S = \frac{1}{2}E_1 - \frac{1}{2}E_2$ and S is a proper convex combination of two other matrices in Ω_s.

It only remains to show that the matrices with one or no middle entries in each row and column are the extreme points of Ω_s. Let $A \in \Omega_s$ which has has one or no middle entry in each row and column. Assume that A can be expressed as a proper convex combination of two other matrices in Ω_s say $A = \theta B + (1 - \theta)C$ where B and C are in Ω_s and $0 < \theta < 1$.

Suppose $a_{ij} = 1/m$ then $\theta b_{ij} + (1 - \theta) c_{ij} = 1/m$ or $\theta(b_{ij} - c_{ij}) = (1/m) - c_{ij}$. This is the same as $\theta(c_{ij} - b_{ij}) = c_{ij} - (1/m)$. When $c_{ij} = 1/m$ then $b_{ij} = 1/m = a_{ij}$. When $c_{ij} > 1/m$ then $c_{ij} - b_{ij} > c_{ij} - (1/m)$ which implies that $b_{ij} < 1/m$ which is impossible. Therefore, if $a_{ij} = 1/m$ then $b_{ij} = c_{ij} = a_{ij} = 1/m$.

Now suppose $a_{ij} = 1/t$, then $\theta b_{ij} + (1 - \theta) c_{ij} = 1/t$ or $\theta(b_{ij} - c_{ij}) = (1/t) - c_{ij}$. When $c_{ij} = 1/t$ then $b_{ij} = 1/t = a_{ij}$. When $c_{ij} < 1/t$ then $b_{ij} - c_{ij} > 1/t = c_{ij}$ which implies $b_{ij} > 1/t$ which is impossible. Therefore, if $a_{ij} = 1/t$ then $b_{ij} = c_{ij} = 1/t = a_{ij}$.

A has $n - 1$ (or n) entries in each row and column equal to $1/m$ or $1/t$ which determines $n - 1$ (or n) entries in each row and column of B and C. A, B, and C are all doubly stochastic which means that if $n - 1$ positions in a row or column is determined then all n positions are determined. Therefore, $A = B = C$ and A cannot be expressed as a proper convex combination of two other matrices in Ω_s.

Reference

1. W. Watkins and R. Merris, Convex sets of doubly stochastic matrices, *J. Combinatorial Theory (A)* **16** (1974), 129–130.